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Towards Realizing Dynamical SUSY Breaking in Heterotic Model Building

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Abstract

We study a new mechanism to dynamically break supersymmetry in the $E_8 \times E_8$ heterotic string. As discussed recently in the literature, a long-lived, meta-stable non-supersymmetric vacuum can be achieved in an $\mathcal{N} = 1$ SQCD whose spectrum contains a sufficient number of light fundamental flavors. In this paper, we present, within the context of the hidden sector of the weakly and strongly coupled heterotic string, a slope-stable, holomorphic vector bundle on a Calabi-Yau threefold for which all matter fields are in the fundamental representation and are massive at generic points in moduli space. It is shown, however, that near certain subvarieties in the moduli space a sufficient number of light matter fields can occur, providing an explicit heterotic model realizing dynamical SUSY breaking. This is demonstrated for the low-energy gauge group $Spin(10)$. However, our methods immediately generalize to $Spin(N_c)$, $SU(N_c)$, and $Sp(N_c)$, for a wide range of color index N_c . Moduli stabilization in vacua with a positive cosmological constant is briefly discussed.

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1 Introduction

Heterotic M -theory [1, 2, 3, 4, 5, 6] provides a promising framework to construct string theory vacua with the spectrum of the supersymmetric standard model. Recently, vacua of this kind were obtained in heterotic compactifications on non-simply connected Calabi-Yau manifolds [7, 8, 9, 10, 11, 12]. One important task is to understand how

supersymmetry can be broken in these models. A natural attempt would be to create a hidden sector with broken supersymmetry and to communicate this breaking to the standard model sector via some mediation mechanism. Recently, a discussion of such mechanisms in various string compactification and brane models was presented in [13]. In [14], Intriligator, Seiberg, and Shih demonstrated that a class of $\mathcal{N} = 1$ SQCD theories generates dynamical SUSY breaking in a metastable vacuum. This class involves theories whose matter spectrum consists solely of N_f massive fundamental multiplets, where N_f is in the free magnetic range. The existence of this vacuum can then be explicitly established by using the Seiberg dual description [15, 16, 17], in which the SUSY breaking vacuum appears at weak coupling. In addition, these theories have N_c supersymmetric vacua so that the SUSY breaking vacuum is metastable.

It is important to understand whether this type of supersymmetry breaking can be embedded in string theory, especially in realistic compactifications and brane models with stable moduli. In [18, 19] these questions were studied in Type II string theory. In [20], we began the study of how dynamical SUSY breaking can be realized in realistic theories of the $E_8 \times E_8$ heterotic string. In this paper, we continue this research, presenting all the requisite technical details and proofs leading to the results in [20]. The obvious approach is to construct vacua whose low-energy field theory satisfies the criteria of [14] in the hidden sector. Since the desired low energy theory is non-chiral, we have to choose a hidden sector vector bundle with vanishing third Chern class. The spectrum of light particles is determined by the cohomology groups with coefficients in different products of this vector bundle. As one moves in the associated moduli space, some of the non-chiral matter becomes massless on higher co-dimension subvarieties. Thus, the first step would be to find a subvariety on which the massless spectrum satisfies the representation and multiplicity criteria of [14]. Then as we move slightly away from this subvariety, the matter receives a small mass, which is the final requirement in [14]. Unfortunately, moduli spaces of Calabi-Yau manifolds and vector bundles are complicated and it is usually difficult to prove the existence of subvarieties with the requisite properties. In this paper, we will explicitly construct one class of examples where this is achieved. The structure group of the vector bundle is chosen to be $SU(4)$, which leads to a low-energy field theory with gauge group $Spin(10)$. We show that, in this example, it is possible to constrain the moduli in such a way that N_f fundamental multiplets of $SO(10)$, for any integer N_f , obtain light masses whereas all other matter fields are heavy and can be integrated out. This gives an example of a class of vacua of the heterotic string whose low-energy field theories satisfying the criteria of [14]. It is important to note that heterotic compactifications potentially have completely stabilized moduli. We will not discuss this in the present paper. Various aspects of moduli stabilization in different heterotic models can be found in [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

This paper is organized as follows. In Section 2, we review the criteria that theories

with dynamical SUSY breaking must satisfy [14]. In Section 3, a general discussion of quadratic superpotentials for matter fields in heterotic compactifications is presented. The fact that the number of light fields changes as we move in the moduli space means that there exists a non-vanishing quadratic superpotential whose mass coefficients are moduli dependent. The generic superpotential is one which is cubic in the open string fields. That is, it is a quadratic function in the matter fields and a linear function in the vector bundle moduli. However, there can also be higher order contributions to the superpotential of open string fields arising from integrating out heavy Kaluza-Klein modes. Additionally, we give a brief discussion of moduli stabilization and the possible relevance of the metastable SUSY breaking for obtaining vacua with a small, positive cosmological constant. In Section 4, it is shown how the matter spectrum can change for special values of the moduli. This is the basis for our choice of hidden sector vector bundle, which we describe in Section 5.

To be as explicit as possible, we study the vector bundle and region of moduli space leading to $N_f = 8$ flavors. The Calabi-Yau threefold is a double elliptic fibration over dP_9 (del Pezzo) surfaces, and the $SU(4)$ vector bundle is constructed as a non-trivial extension with building blocks pulled back from the two different dP_9 bases. The slope-stability of the vector bundle is proven in Subsection 5.3, and follows from well-known results about extensions of spectral cover bundles. This choice of the vector bundle makes the study of cohomology groups tractable. On each dP_9 surface, the cohomology groups of interest are localized at points. The dimensions of the cohomology groups becomes the number of points where the supports of two such factors overlap. We also show that the superpotential is cubic in the open string fields. By arranging the supports in the right way, one can construct $\mathcal{N} = 1$, $SO(N_c)$ SQCD with N_f massive fundamentals satisfying the criteria of [14]. Hence, one can construct vector bundles of this type where the number of light fundamental representations lies in the appropriate range for dynamical symmetry breaking.

In Section 6, we present a mathematical proof of the various details of the spectral cover used in Section 5. In the conclusion, Section 7, we discuss possible extensions of our results. Finally, mathematical properties of the support of line bundles and derived tensor products needed in our analysis are presented in Appendices A and B respectively.

2 Dynamical SUSY Breaking

In this section, we will give a brief review of dynamical supersymmetry breaking following [14]. We will state certain necessary ingredients which will be used in later sections.

The main example studied in [14] was $\mathcal{N} = 1$, $SU(N_c)$ SQCD with N_f fundamental

flavors Q, \tilde{Q} in the free magnetic range [15, 16]

$$N_c + 1 \leq N_f < \frac{3}{2}N_c. \quad (2.1)$$

The flavors have a quadratic superpotential

$$W = \text{Tr } m M, \quad (2.2)$$

where

$$M = Q_f \cdot \tilde{Q}_g, \quad f, g = 1, \dots, N_f, \quad (2.3)$$

so that they are all massive. This theory is known to have N_c supersymmetric vacua with

$$\langle M \rangle = (\Lambda^{3N_c - N_f} \det m)^{1/N_c} m^{-1}, \quad (2.4)$$

where Λ is the strong-coupling scale. It was shown in [14] that, in addition, this theory has a metastable SUSY breaking vacuum. This was established by studying the Seiberg dual [15, 16] of the original theory. The Seiberg dual theory is $SU(N_f - N_c)$ SQCD with N_f fundamentals q, \tilde{q} and N_f^2 extra singlets Φ_f^q . It has a quadratic leading order Kähler potential and the superpotential is given by (up to some field redefinition)

$$W_{\text{dual}} = h \text{Tr } q \Phi \tilde{q} - h \mu^2 \text{Tr } \Phi \quad (2.5)$$

where $\mu = \sqrt{m\Lambda}$ and h is a dimensionless parameter defined in [14]. For simplicity, we have assumed that all eigenvalues of the mass matrix are equal. This theory breaks supersymmetry by a rank condition mechanism since F-flatness for Φ requires that

$$\tilde{q}^f q_g = \mu^2 \delta_g^f, \quad (2.6)$$

which cannot be satisfied because the number of colors of the dual theory $N_f - N_c$ is less than the number of flavors N_f . However, it was shown in [14] that there exists a metastable SUSY breaking vacuum with

$$V_{\min} = N_c |h^2 \mu^4|, \quad (2.7)$$

a result which can be trusted in the regime

$$\epsilon \sim \sqrt{\left| \frac{m}{\Lambda} \right|} \ll 1. \quad (2.8)$$

Furthermore, as $\epsilon \rightarrow 0$ this state becomes very long-lived. For ϵ sufficiently small, the life-time of the meta-stable state can exceed the age of the Universe, making these vacua of phenomenological interest.

These results were also generalized in [14] for SQCD with gauge groups $SO(N_c)$ and $Sp(N_c)$. In this paper, we will be particularly interested in $SO(N_c)$ theories. Hence we

review some important facts about them [14, 17]. $SO(N_c)$ SQCD has only one type of fundamental representation Q_f . The tree-level superpotential is given by eq. (2.2) with

$$M = Q_f \cdot Q_g. \quad (2.9)$$

The free magnetic range is defined by

$$N_c - 2 < N_f < \frac{3}{2}(N_c - 2). \quad (2.10)$$

The Seiberg dual theory then has the (non-Abelian) gauge group $SO(N_f - N_c + 4)$ and the tree-level superpotential of the type eq. (2.5) with \tilde{q} replaced by q . For $N_f = N_c - 2$, the Seiberg dual gauge group is $SO(2) \simeq U(1)$. Thus, the dual theory is really in the Coulomb phase. However, the SUSY breaking vacuum still exists. Finally, there are special cases for $N_f = N_c - 3$ and $N_f = N_c - 4$. A detailed investigation [14] reveals that they have SUSY breaking vacua as well.

To summarize, if the number of fundamentals is in the range

$$N_c - 4 \leq N_f < \frac{3}{2}(N_c - 2), \quad (2.11)$$

then the $SO(N_c)$ theory has a metastable SUSY breaking vacuum, which can be trusted in the regime eq. (2.8). Details of dynamical SUSY breaking in SQCD with the gauge group $Sp(N_c)$ can be found in Subsection 6.3 of [14].

3 Embedding in Heterotic Compactifications

Compactifications of heterotic string theory or heterotic M-theory provide a promising way of obtaining a realistic supersymmetric standard model spectrum with stabilized moduli. The models of [14], reviewed in the previous section, can provide a mechanism to break supersymmetry in heterotic compactifications. Below we will give a general discussion of how dynamical supersymmetry breaking can be embedded in heterotic compactifications as the hidden sector. In the next section, we will present a concrete class of heterotic compactifications where the spectrum satisfies the requisite properties of [14].

3.1 Quadratic Superpotentials for Matter Fields

An important ingredient of dynamical SUSY breaking models is the tree level quadratic superpotential. Therefore, it is important to discuss how quadratic superpotentials for matter field can arise in heterotic compactifications. Let X be a compactification Calabi-Yau threefold and V be a vector bundle. The massless particle spectrum is associated with the zero modes of the Dirac operator on X . Such zero modes are in one-to-one

correspondence with bundle-valued closed differential $(0, 1)$ -forms and, hence, bundle cohomology groups $H^1(X, U)$, where U can be $V, V^\vee, \wedge^2 V, \dots$. Cohomology groups with coefficients in different U bundles define the massless states in the corresponding representations of the low-energy gauge group in four dimensions.

However, the dimensions of these bundle cohomology groups are not a topological invariant. They depend on the location in the vector bundle and complex structure moduli space. As we move in these moduli spaces, $h^1(X, U)$ can jump. This means that the corresponding four-dimensional fields have a quadratic superpotential with the mass depending on the vector bundle and complex structure moduli. Somewhere in the moduli space these masses can vanish, thus increasing the number of the massless particles. If a compactification has some chiral matter, then a certain number of fields will always stay massless since they are protected by a topological invariant, the Atiyah-Singer index. On the other hand, the models reviewed in Section 2 are non-chiral. Hence, we are interested in compactifications with no chiral matter. In this case, there are no obvious obstructions to all matter multiplets having a non-vanishing potential. One should expect, in compactifications with no chiral matter, that every matter field will have a quadratic potential at a generic point in moduli space. However, as we move in the moduli space some fields can become light on higher co-dimension subvarieties.

Let us now discuss where quadratic potentials for matter fields can come from. Let Q be a four-dimensional matter field transforming in some representation R of the low-energy gauge group, \tilde{Q} be a matter field in the conjugate representation \bar{R} (\tilde{Q} might coincide with Q if R is real) and ϕ represent vector bundle moduli. All these fields correspond to $(0, 1)$ -forms on X with coefficients in the vector bundles U_R, U_R^\vee and $\text{ad}(V)$ respectively. Denote these forms as $\Psi_Q, \Psi_{\tilde{Q}}$ and Ψ_ϕ . Upon dimensional reduction, these fields get a cubic superpotential (see, for example, [36]) of the form

$$W = \lambda \phi \text{Tr } Q \tilde{Q}. \quad (3.1)$$

The coefficients λ depend on the complex structure and vector bundle moduli and are given by

$$\lambda = \int_X \Omega \wedge \text{Tr} \left(\Psi_\phi \wedge \Psi_Q \wedge \Psi_{\tilde{Q}} \right), \quad (3.2)$$

where Ω is the holomorphic $(3, 0)$ -form. If Q and \tilde{Q} are Higgs fields, this superpotential represents a μ -term for them. Recently, such μ -terms were computed in realistic compactification scenarios in [37, 38, 39]. The superpotential eq. (3.1) provides a generic mechanism for non-chiral matter to receive a mass depending on various moduli. In addition, the open string fields can also get a quartic superpotential of the form

$$W \sim \phi \phi \text{Tr } Q \tilde{Q}. \quad (3.3)$$

Such a superpotential can arise after integrating out massive Kaluza-Klein modes. Indeed, let \tilde{Q}_{KK} be a Kaluza-Klein mode in the representation \bar{R} . It can couple to ϕ

and Q through the a superpotential similar to eq. (3.1). In addition, it has a quadratic superpotential with constant mass of order the compactification scale. Integrating \tilde{Q}_{KK} out is equivalent to eliminating auxiliary fields in supersymmetric field theories. This procedure yields a quartic superpotential of the form eq. (3.3).

3.2 On Moduli Stabilization and Vacua with a Positive Cosmological Constant

Eventually, we are interested in compactifications with stable moduli. In this case, we can replace the complex structure and vector bundle moduli with their vacuum expectation values (VEV), thus obtaining a quadratic superpotential for the non-chiral matter. Let us consider a compactification leading, at low energy, to a heterotic standard model in the observable sector and to a hidden sector with gauge group $SU(N)$, $SO(N)$, or $Sp(N)$. As an example, one can take the structure group of the hidden sector vector bundle to be $SU(5)$, thus obtaining another $SU(5)$ as the low energy gauge group. Another example is to choose an $SU(4)$ structure group, leading to an $SO(10)$ low energy gauge group in the hidden sector. We start our analysis by ignoring all couplings to matter fields and finding a supersymmetric AdS vacuum by solving

$$D_{\text{moduli}} W_{\text{moduli}} = 0. \quad (3.4)$$

Questions concerning moduli stabilisation in heterotic compactifications were studied in [21, 22, 23, 40, 41, 24, 25, 26, 27, 28, 29, 30, 31, 32], and we will not review them in this paper. For our purposes, we assume that eq. (3.4) stabilizes all the moduli in a phenomenologically acceptable range. We further assume that the moduli VEVs give the hidden sector fundamental matter, for all N_f flavors in the free magnetic range, a small mass from the superpotential eq. (3.1). All the remaining non-chiral matter has very heavy mass and is integrated out. In the next section, we will present an explicit example of a compactification with such properties. By the results of [14], the supersymmetry will then be broken dynamically in the hidden sector. This supersymmetry breaking is communicated to the standard model sector by one of the mediation mechanisms (see, for example, [42] for a review). The metastable SUSY breaking vacuum obtained in [14] can be trusted in the regime where $\epsilon \ll 1$ and where one can neglect the $\frac{1}{M_{Pl}}$ contributions to the potential energy. This dynamical SUSY breaking also has an obvious effect on the cosmological constant. Let W_0 be the value of the moduli superpotential in the solution eq. (3.4). It produces a negative contribution to the cosmological constant of order $-3\frac{|W_0|^2}{M_{Pl}^2}$. On the other hand, the matter in the hidden sector in the metastable SUSY breaking vacuum gives a positive correction to the cosmological constant. Depending on the relative values of m and W_0 , one can obtain a non-supersymmetric vacuum with a negative, vanishing, or positive cosmological constant. In particular, vacua with a small,

positive cosmological constant can potentially be obtained this way. This important physics is model dependent and goes beyond the range of this paper. Hence, we will not discuss it here but leave it for future research.

4 Mass Terms and Discontinuous Cohomology

Before we are going to delve into the technicalities of our model, let us first describe the underlying idea of the construction. As described in Section 3, we want a hidden sector that contains no massless matter fields at a generic point in the moduli space, but does contain massless matter for special values of the moduli. This is possible since the sheaf cohomology that computes the spectrum is not a topological invariant, but can in fact change as one varies the vector bundle moduli [43, 44, 45, 46]. The simplest such “jump” occurs already for an elliptic curve.

Let us start by reviewing this case, and take

$$E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \quad (4.1)$$

to be an elliptic curve, and let us fix the point $0 + 0i = o \in E$. An elliptic curve with origin is, in fact, a group: The group law \boxplus on the points of E is addition in \mathbb{C} modulo the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Now the divisors on E are formal \mathbb{Z} -linear combinations of points, and every line bundle can be written as

$$\mathcal{O}_E\left(\sum_{i=1}^n p_i - \sum_{j=1}^m q_j\right), \quad p_i, q_j \in E \quad (4.2)$$

But not all such line bundles are distinct, and the isomorphism classes of holomorphic line bundles on E can be labeled by the two numbers

$$n - m \in \mathbb{Z} \quad (4.3a)$$

$$\left(\boxplus_{i=1}^n p_i\right) \boxminus \left(\boxplus_{j=1}^m q_j\right) \in E \quad (4.3b)$$

Depending on these two invariants, there are four cases to distinguish. They are

| | Case 1 | Case 2 | Case 3 | Case 4 | |
|---|---------|-----------------|--------|---------|-------|
| $n - m$ | > 0 | $= 0$ | $= 0$ | < 0 | |
| $\left(\boxplus_{i=1}^n p_i\right) \boxminus \left(\boxplus_{j=1}^m q_j\right)$ | any | $\in E - \{o\}$ | $= o$ | any | (4.4) |
| $\dim H^0\left(\sum_{i=1}^n p_i - \sum_{j=1}^m q_j\right)$ | $n - m$ | 0 | 1 | 0 | |
| $\dim H^1\left(\sum_{i=1}^n p_i - \sum_{j=1}^m q_j\right)$ | 0 | 0 | 1 | $m - n$ | . |

In particular, we are interested in the $n - m = 0$ case. Then the line bundle has vanishing first Chern class, but there are still two possibilities. Either the line bundle is the trivial

line bundle $\mathcal{O}_E = E \times \mathbb{C}$, or the line bundle is of the form $\mathcal{O}_E(p - o)$ for some $p \neq o \in E$. In the first case $H^0(E, \mathcal{O}_E) = H^1(E, \mathcal{O}_E) = \mathbb{C}$, while in the latter case all cohomology groups vanish.

The underlying idea of the spectral cover construction is to apply this fiberwise to an elliptic fibration. Consider a spectral curve C that is a k -fold cover of the base, and let σ be the zero section of the elliptic fibration. Then C intersects a generic fiber f in k separate points C_1, \dots, C_k , and σ intersects the fiber f in the single point $o \in F$. The Fourier-Mukai transform constructs a rank k vector bundle whose restriction to f is

$$\mathrm{FM}(\mathcal{O}_C)|_f = \mathcal{O}_f(C_1 - o) \oplus \dots \oplus \mathcal{O}_f(C_k - o). \quad (4.5)$$

Obviously, the cohomology of $\mathrm{FM}(\mathcal{O}_C)|_f$ vanishes unless one of the points C_1, \dots, C_k coincides with o . But according to the Leray spectral sequence (see, for example, [47]), the cohomology groups of $\mathrm{FM}(\mathcal{O}_C)$ can be computed in terms of the fiberwise cohomology. If the latter vanishes, then the cohomology of $\mathrm{FM}(\mathcal{O}_C)$ has to vanish as well.

Note that it is not enough if only the cohomology at generic fibers vanishes, but only if it vanishes at every fiber. Since the intersection points $C \cap f$ vary as we vary the fiber f , we expect that there are some fibers where $C_1 = o$ or $C_2 = o$ or \dots or $C_k = o$. In terms of the zero section σ of the elliptic fibration, these points are $C \cdot \sigma$. As $C_i = o$ is one complex equation, these special fibers occur in codimension one on the base. If we were to consider an elliptically fibered Calabi-Yau threefold, then the complex 2-dimensional base will in general contain a curve which supports cohomology groups. Instead, we will take the Calabi-Yau threefold X to be fibered over \mathbb{P}^1 ,

$$X \xrightarrow{pr} \mathbb{P}^1, \quad (4.6)$$

such that a generic fiber

$$pr^{-1}(\{\mathrm{pt.}\}) \simeq E_1 \times E_2 \quad (4.7)$$

factors into the product of two elliptic curves. Then we arrange spectral cover-like bundles on E_1 and E_2 separately, that is, construct a bundle such that the restriction to $pr^{-1}(\{\mathrm{pt.}\})$ is

$$\left(\mathcal{O}_{E_1}(C_1 - o) \oplus \dots \oplus \mathcal{O}_{E_1}(C_k - o) \right) \boxtimes \left(\mathcal{O}_{E_2}(D_1 - o) \oplus \dots \oplus \mathcal{O}_{E_2}(D_l - o) \right), \\ C_1, \dots, C_k \in E_1, \quad D_1, \dots, D_l \in E_2. \quad (4.8)$$

Generically none of the points $\{C_1, \dots, C_k\}$ and none of the points $\{D_1, \dots, D_l\}$ coincides with o , and the cohomology along the fiber direction vanishes. Only if $C_i = o = D_j$ simultaneously for some $i = 1, \dots, k, j = 1, \dots, l$ then the cohomology of the bundle eq. (4.8) is non-vanishing. But that yields two complex equations on the 1-dimensional

base \mathbb{P}^1 , which has no solutions in general. Only specially designed bundles then have non-zero cohomology groups, while any small deformation will lead to vanishing cohomology.

5 The Compactification

5.1 The Calabi-Yau Threefold

In this section, we will present a concrete model of the hidden sector satisfying the criteria for dynamical SUSY breaking. Since we are only interested in the supersymmetry breaking in the hidden sector, we will not specify the visible sector and the five-brane structure. In our model, we choose the Calabi-Yau threefold X to be a double elliptic fibration [48, 49, 50, 51, 52]

$$X = B_1 \times_{\mathbb{P}^1} B_2, \quad (5.1)$$

where

$$B_1 \simeq \text{dP}_9, \quad B_2 \simeq \text{dP}_9 \quad (5.2)$$

are two rational elliptic (dP_9) surfaces. We will denote projections by $\pi_i = X \rightarrow B_i$ and $\beta_i = B_i \rightarrow \mathbb{P}^1$, $i = 1, 2$, yielding a commutative square

$$\begin{array}{lcl} \dim_{\mathbb{C}} = 3 : & & X \\ & \swarrow \pi_1 & \searrow \pi_2 \\ \dim_{\mathbb{C}} = 2 : & B_1 & B_2 \\ & \searrow \beta_1 & \swarrow \beta_2 \\ \dim_{\mathbb{C}} = 1 : & & \mathbb{P}^1. \end{array} \quad (5.3)$$

The fibers of these projections are generically elliptic curves, with some degenerate fibers. The Abelian surface fibration of Section 4, eq. (4.7) is simply $pr = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$. Let us state some properties of the homology group of curves $H_2(B_i, \mathbb{Z})$ which we will be using. A dP_9 surface is obtained by blowing up nine points of \mathbb{P}^2 . From \mathbb{P}^2 we inherit the class of the hyperplane divisor ℓ , and each blow-up adds one exceptional divisor. Hence¹,

$$H_2(B_i, \mathbb{Z}) = 1 + 9 = 10. \quad (5.4)$$

We denote these 9 exceptional divisors e_i , $i = 1, \dots, 9$. The intersection numbers of these classes are

$$\ell \cdot \ell = 1, \quad e_i \cdot e_j = -\delta_{ij}, \quad e_i \cdot \ell = 0. \quad (5.5)$$

¹The construction involves two distinct dP_9 surfaces B_1 and B_2 . Hence, strictly speaking, one needs to distinguish their divisors by labeling them differently. However, it will always be clear from the context which surface we are referring to. Therefore, we suppress this extra label.

Obviously the determinant of the intersection matrix is -1 , and therefore the classes $\ell, \epsilon_1, \dots, \epsilon_9$ are an integral basis for the homology lattice. In this basis the fiber class of the dP_9 elliptic fibration reads

$$f = 3\ell - \sum_{i=1}^9 e_i. \quad (5.6)$$

Each exceptional divisor e_i is a section of dP_9 since it intersects the fiber f at one point. We will choose e_9 to be the zero section.

Finally, we need the even cohomology ring to compute Chern classes. It is generated by the pull-backs

$$\begin{aligned} \lambda^1 &= \pi_1^*(\ell), & \epsilon_i^1 &= \pi_1^*(e_i), & i &= 1, \dots, 9, \\ \lambda^2 &= \pi_2^*(\ell), & \epsilon_i^2 &= \pi_2^*(e_i), & i &= 1, \dots, 9, \end{aligned} \quad (5.7)$$

see [49, 53, 9]. The T^4 fiber can be expressed in two different ways, yielding the relation

$$3\lambda^1 - \sum_{i=1}^9 \epsilon_i^1 = 3\lambda^2 - \sum_{i=1}^9 \epsilon_i^2. \quad (5.8)$$

In addition, there are quadratic relations that are inherited from the base dP_9 surfaces

$$\begin{aligned} (\lambda^1)^2 &= -(\epsilon_1^1)^2, & (\lambda^2)^2 &= -(\epsilon_1^2)^2, \\ \lambda^1 \epsilon_i^1 &= 0, & \lambda^2 \epsilon_i^2 &= 0, & i &= 1, \dots, 9, \\ \epsilon_i^1 \epsilon_j^1 &= \delta_{ij} (\epsilon_1^1)^2, & \epsilon_i^2 \epsilon_j^2 &= \delta_{ij} (\epsilon_1^2)^2, & i, j &= 1, \dots, 9, \end{aligned} \quad (5.9)$$

and one set of relations that involves both dP_9 surfaces,

$$(\epsilon_i^1 - \epsilon_j^1)(\epsilon_k^2 - \epsilon_l^2) = 0, \quad i, j, k, l = 1, \dots, 9. \quad (5.10)$$

To summarize, the even cohomology groups are

$$H^{\text{ev}}(X, \mathbb{Z}) = \mathbb{Z}[\lambda^1, \lambda^2, \epsilon_1^1, \dots, \epsilon_9^1, \epsilon_1^2, \dots, \epsilon_9^2] / \{\text{Relations eqns. (5.8), (5.9), (5.10)}\}. \quad (5.11)$$

5.2 The Vector Bundle

Having described the base space X , we now construct a slope-stable, holomorphic vector bundle V with structure group $SU(4)$ and vanishing third Chern class. Turning on such an instanton in the hidden sector E_8 gauge group breaks it to $Spin(10)$. There are two types of matter fields that appear in four dimensions, one can have multiplets transforming as **16**, **$\overline{16}$** , and **10** of $Spin(10)$. Their number is given by $h^1(X, V)$, $h^1(X, V^\vee)$,

and $h^1(X, \wedge^2 V)$, respectively. Since we chose the third Chern class of V to be zero, the number of 16 and the number of 16 is the same.

Let us now describe the vector bundle V . We construct the rank 4 vector bundle V as a non-trivial extension of a line bundle and a rank 3 vector bundle, that is,

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_3 \longrightarrow 0. \quad (5.12)$$

The rank 3 bundle V_3 will be

$$V_3 = \pi_1^*(L) \otimes \pi_2^*(W), \quad (5.13)$$

where L is a line bundle on B_1 and W is a rank 3 vector bundle on B_2 defined as follows. The line bundle is

$$L = \mathcal{O}_{B_1}(e_1 - e_9). \quad (5.14)$$

Really we could use the difference of any two sections that do not intersect, but for definiteness we will use the exceptional divisors e_1 and e_9 .

Furthermore, we define the rank three vector bundle W using the spectral cover construction. The spectral curve C_W , see [54], is taken to be an irreducible curve in the linear system

$$C_W \in \Gamma \mathcal{O}_{B_2}(\ell + f). \quad (5.15)$$

From eqns. (5.5) and (5.6) it follows that ℓ intersects f at three points and, thus, is a triple cover of the base \mathbb{P}^1 . In addition to the spectral curve C_W , we also have to specify a line bundle N_W on C_W . For simplicity, we take N_W to be the trivial line bundle on C_W

$$N_W = \mathcal{O}_C. \quad (5.16)$$

The stable rank 3 vector bundle W is then obtained by the Fourier-Mukai transform of (C_W, N_W) [54, 55],

$$W = \text{FM}_{B_2}(\mathcal{O}_C). \quad (5.17)$$

Using the action of the Fourier-Mukai transform on the level of Chern classes which were worked out in [56], we find

$$\text{rank}(W) = 3, \quad c_1(W) = \ell - 3e_9 - 8f, \quad c_2(W) = 0; \quad (5.18)$$

Note that W is a $U(3)$ vector bundle with non-trivial $U(1)$ part

$$\det W = \mathcal{O}_{B_2}(\ell - 3e_9 - 8f) \quad (5.19)$$

In particular, $\wedge^2 W$ is not isomorphic to W^\vee .

Using again techniques developed in [56], one finds that the spectral cover of $\wedge^2 W$ is in the linear system

$$C_{\wedge^2 W} \in \Gamma \mathcal{O}_{B_2}(-2\ell + 9e_9 + 14f). \quad (5.20)$$

To make sure that V has structure group $SU(4)$, we finally pick the line bundle V_1 to be

$$V_1 = \pi_1^*(L^{-3}) \otimes \pi_2^*(\det^{-1} W). \quad (5.21)$$

This choice of V_1 guarantees that the first Chern class of V vanishes.

5.3 Chern Classes and Stability

Knowing the even cohomology ring explicitly eq. (5.11), one can easily compute all relevant Chern classes. One finds

$$\text{rank}(V) = 4, \quad c_1(V) = 0, \quad c_2(V) = 12(\lambda^1)^2 + 8(\lambda^2)^2, \quad c_3(V) = 0. \quad (5.22)$$

Therefore, the gauge and gravity contribution to the heterotic anomaly equation for some visible sector bundle V_{vis} reads

$$\begin{aligned} c_2(TX) - c_2(V_{\text{vis}}) - c_2(V) &= \\ &= \left(12(\lambda^1)^2 + 12(\lambda^2)^2\right) - \left(12(\lambda^1)^2 + 8(\lambda^2)^2\right) - c_2(V_{\text{vis}}) = \\ &= 4(\lambda^2)^2 - c_2(V_{\text{vis}}), \end{aligned} \quad (5.23)$$

where $(\lambda^i)^2$ is the fiber class of π_i , $i = 1, 2$, and hence an effective curve. We conclude that, depending on V_{vis} , it is possible to cancel the heterotic anomaly without introducing anti-five-branes.

To show that V is slope-stable for some suitable Kähler class, we only have to satisfy [57, 49]

1. The extension eq. (5.12) is not split.
2. The slope of V_1 is negative.

We are going to compute the extensions in Subsection 5.5, and find that there are non-trivial extensions. Finally, if one prefers to work in a region of the Kähler moduli space where the slope $\mu(V_1)$ of V_1 is positive, then

$$\mu(V_1) > 0 \quad \Leftrightarrow \quad \mu(V_3) < 0. \quad (5.24)$$

In that case, one can just reverse the extension eq. (5.12). It turns out that for our bundle V this does not influence any cohomology groups. Hence, in one way or the other V is slope-stable.

5.4 Localization of Cohomology

In this subsection, we will review some basics of sheaf cohomology and how it applies to the vector bundles V and $\wedge^2 V$ which we are using throughout this paper. A detailed consideration of them in a similar geometry can be found, for example, in subsections 7.3, 7.4 of [44]. Let

$$X \xrightarrow{\pi} B \quad (5.25)$$

be an elliptically fibered manifold, and U be a slope-stable, holomorphic vector bundle obtained via the spectral cover construction [54, 55]. In particular, we assume that the restriction $U|_F$ to a generic fiber is regular semistable and of degree 0. Applying the Leray spectral sequence to that case (see, for example, [47]), one finds that the cohomology groups with coefficients in U are localized in a codimension two subvariety of X . As we saw in Section 4, the localized cohomology groups correspond to the intersection points

$$C_U \cdot \sigma, \quad (5.26)$$

where C_U is the spectral cover² and σ is the zero section. In our case the Leray spectral sequence determines the cohomology groups of U on X in terms of the cohomology of certain torsion sheaves on B with support on $\pi(C_U \cdot \sigma)$. This torsion sheaf happens to be

$$R^1 \pi_* U, \quad (5.27)$$

the sheaf whose “fiber” over a point $p \in B$ is $H^1(f_p, U|_{f_p})$, where $f_p = \pi^{-1}(p)$ is the fiber at the point p . Hence,

$$H^1(X, U) = H^0(B, R^1 \pi_* U). \quad (5.28)$$

As we discussed in Section 4, the “fiber” dimension of $R^1 \pi_* U$ is generically zero but can jump occasionally, hence the $R^1 \pi_* U$ is only a coherent sheaf and not a vector bundle.

In our case, the building blocks of the bundle V are vector bundles on the dP_9 surfaces. In the following, it will be useful to specialize the above to the case where the total space is the surface B_i with projection $\beta_i : B_i \rightarrow \mathbb{P}^1$. Let us denote the spectral curve C_{U_i} and the corresponding bundle U_i . In that case, $C_{U_i} \cdot \sigma$ consists of a certain number of points. The sheaf $R^1 \beta_{i*} U_i$ is the skyscraper sheaf supported at these points and zero everywhere else. At each of these points $R^1 \beta_{i*} U_i$ is just \mathbb{C} . Thus³,

$$H^1(B_i, U_i) = H^0(\mathbb{P}^1, R^1 \beta_{i*} U_i) = H^0(C_U \cdot \sigma, \mathbb{C}) \quad (5.29)$$

²The result for $\wedge^2 U$ is identical with C_U being replaced by the spectral cover $C_{\wedge^2 W}$ of $\wedge^2 W$.

³Here we are assuming for simplicity that the zero section σ does not meet any singularities of the spectral curve C_U . If they do meet in singular points it is still true that the cohomology is supported at these points, one just has to be careful with the multiplicities.

is nothing else than the number of points where the spectral cover intersects the zero section. All higher cohomology vanish, and the details of $R^1\beta_{i*}U_i$ become irrelevant. Let us denote by $\text{Hsupp}(U_i)$ the points where the cohomology of U_i is supported, that is,

$$\text{Hsupp}(U_i) = \text{supp}(R^1\beta_{i*}U_i) = \beta_i(C_{U_i} \cap \sigma) \subset \mathbb{P}^1. \quad (5.30)$$

In the next section, we will be interested in a special limit where points in $\text{Hsupp}(U_i)$ collide. For that case one has to count the points with multiplicities. Finally, we remark that

$$\text{Hsupp}(U_i^\vee) = \beta_i((\boxminus C_{U_i}) \cap \sigma) = \beta_i(C_{U_i} \cap \sigma) = \text{Hsupp}(U_i). \quad (5.31)$$

In this paper, we will often encounter the case where the bundle on the threefold $X = B_1 \times_{\mathbb{P}^1} B_2$ is the tensor product of bundles pulled back from B_1 and B_2 , that is,

$$U = \pi_1^*(U_1) \otimes \pi_2^*(U_2) \quad (5.32)$$

Such a vector bundle is, when restricted to a T^4 -fiber of the fibration $pr = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2$, of the form eq. (4.8). Hence the discussion at the end of Section 4 applies, and we expect U to have no cohomology for generic values of the moduli. Its cohomology can be computed by applying the Leray spectral sequence twice, pushing down from X via B_i to \mathbb{P}^1 . One obtains⁴

$$\begin{aligned} H^2(X, U) &= \left\{ \begin{array}{l} H^1\left(U_1 \otimes \beta_1^* \circ R^1\beta_{2*}(U_2)\right) \\ H^1\left(\beta_2^* \circ R^1\beta_{1*}(U_1) \otimes U_2\right) \end{array} \right\} = \\ &= H^0\left(R^1\beta_{1*}(U_1) \otimes R^1\beta_{2*}(U_2)\right) \end{aligned} \quad (5.33)$$

where we either push down via B_1 or B_2 . Obviously, the cohomology of U is supported at the intersection $\text{Hsupp}(U_1) \cap \text{Hsupp}(U_2)$. If $\text{Hsupp}(U_1)$ is distinct from $\text{Hsupp}(U_2)$, we immediately get

$$R^1\beta_{1*}(U_1) \otimes R^1\beta_{2*}(U_2) = 0 \quad \Rightarrow \quad H^2(X, U) = 0. \quad (5.34)$$

In general, $h^1(X, U)$ is given by the number of points common to both supports, that is (counted with appropriate multiplicities),

$$\dim H^2(X, U) = |\text{Hsupp}(U_1) \cap \text{Hsupp}(U_2)|. \quad (5.35)$$

Finally, take the index

$$\chi(U) = \sum_{i=0}^3 H^i(X, U) \quad (5.36)$$

⁴For technical reasons, we compute $H^2(X, U)$. This is explained in Appendix B.

to be zero. This is always the case here, since we construct bundles whose cohomology groups vanish at a generic point in the moduli space. The index is unchanged as one changes the moduli, so if $H^2(X, U)$ jumps then $H^1(X, U)$ has to jump as well to compensate,

$$H^1(X, U) \simeq H^2(X, U) = |\text{Hsupp}(U_1) \cap \text{Hsupp}(U_2)|. \quad (5.37)$$

Here we used that $H^0(X, U) = 0 = H^3(X, U)$ as required by stability.

Let us now apply these results to calculating cohomology $H^1(V)$. In order to reproduce the theory reviewed in Section 2, we should get $H^1(V) = 0$. From the long exact sequence of cohomology associated with the sequence eq. (5.12), we find that $H^1(V) = 0$ if $H^*(V_1) = H^*(V_3) = 0$. Let us show that this is indeed the case at a generic point in the moduli space. Let us start with $H^*(V_3)$. Since the definition of V_3 in eq. (5.13) is of the form eq. (5.32), we can simply apply the previous discussion. One finds that the support of $H^*(V_3)$ is

$$\text{Hsupp}(L) \cap \text{Hsupp}(W). \quad (5.38)$$

Using eqns. (5.5), (5.6) and (5.15), we find that the support of the cohomology of W , $\text{Hsupp}(W)$, is given by

$$C_W \cdot e_9 = 1 \quad (5.39)$$

The precise location of this point depends on the moduli of W . To obtain $\text{Hsupp}(L)$ we have to calculate the sheaf $R^1\beta_{1*}L$. This is performed in Appendix A, and the result is

$$R^1\beta_{1*}L = 0 \quad \Rightarrow \quad \text{Hsupp}(L) = \emptyset \quad (5.40)$$

Thus, $H^*(V_3)=0$. The cohomology of V_1 is supported at

$$\text{Hsupp}(L^3) \cap \text{Hsupp}(\det W). \quad (5.41)$$

One can show that neither support in eq. (5.41) is empty. However, for generic choice of the complex structure of dP_9 and the bundle moduli of W the intersection is empty. Thus, at a generic point in the moduli space $H^*(V_1)$ vanishes and so does $H^*(V)$. This is not surprising. As discussed before, in models with no chiral matter one can expect that all matter has a quadratic superpotential. Thus we can achieve that all particles transforming as 16 and $\overline{16}$ receive a large mass and are integrated out.

5.5 Extensions and the Spectrum of Fundamentals

We want V to be a non-trivial extension. For this we need

$$\text{Ext}^1(V_3, V_1) \neq 0. \quad (5.42)$$

This is equivalent to

$$H^1(X, V_1 \otimes V_3^\vee) = H^1(X, \pi_1^*(L^{-4}) \otimes \pi_2^*(W^\vee \otimes \det^{-1} W)) \neq 0. \quad (5.43)$$

To apply the discussion in the previous subsection we have to understand the intersection of $\text{Hsupp}(L^4)$ and $\text{Hsupp}(W^\vee \otimes \det^{-1} W)$. Despite the fact that the cohomology of L has vanishing support, the line bundle L^2 has non-trivial cohomology. In Appendix A, it is shown that $\text{Hsupp}(L^2)$ consists of three points on \mathbb{P}^1 . Let us denote them by q_1, q_2, q_3 . That is,

$$\text{Hsupp}(L^2) = \{q_1, q_2, q_3\}. \quad (5.44)$$

The actual location of these points depends on the complex structure of X . Furthermore, $\text{Hsupp}(L^4)$ contains fifteen points. It can be shown that these fifteen points contain q_1, q_2, q_3 each with multiplicity one plus 12 other points whose location is completely generic. Let us denote these points by

$$\text{Hsupp}(L^4) = \{q_1, q_2, q_3, s_1, s_2, \dots, s_{12}\}. \quad (5.45)$$

Later in this section we will need to know the cohomology support of the bundle

$$\wedge^2 W = W^\vee \otimes \det W. \quad (5.46)$$

According to our discussion in the previous subsection, it is given by $C_{\wedge^2 W} \cdot e_9$. Using eqns. (5.5), (5.6), and (5.20) these curves intersect in 5 points. The location of these points depends on the moduli of W and the complex structure of X . In the next section we will explicitly demonstrate that there exist a regime in the moduli space where two points appear with multiplicity two. For purposes that will be clear later on, we want to work in this case where

$$\text{Hsupp}(\wedge^2 W) = \{2p_1, 2p_2, p_3\} \quad (5.47)$$

for some points $p_1, p_2, p_3 \in \mathbb{P}^1$. Later, in this subsection, we will see that this choice leads to the spectrum with the number of fundamentals N_f equal to eight. We found that it is the most instructive to do this case in detail. In the next subsection, we will discuss how different numbers of flavors can be obtained. In fact, some of them will be found as a simple modification of the $N_f = 8$ case. Finally, consider $W^\vee \otimes \det^{-1} W = (W \otimes \det W)^\vee$. A quick Chern class computation yields that the cohomology is supported at 21 points. Upon closer inspection in Subsection 6.1, we will see that two of these points are p_1 and p_2 again, leaving us with 19 other points which we denote as

$$\text{Hsupp}(W^\vee \otimes \det^{-1} W) = \{p_1, p_2, r_1, r_2, \dots, r_{19}\}. \quad (5.48)$$

Note that the points q_i, s_j give the cohomology support of bundles on B_1 , and p_k, r_l give the cohomology support of bundles on B_2 . For random values of the moduli,

these two sets of points will be disjoint, and all cohomology groups (including the Ext^1) vanish according to eq. (5.37). Obviously, we want to align some of the points to have extensions and a suitable matter spectrum. Now the actual position of these points depends on complex structure and vector bundle moduli, and one expects to be able to align as many points as there are moduli to adjust. But actually proving this would be cumbersome. Instead, we observe that one can always adjust 3 points by the way our Calabi-Yau threefold $X = B_1 \times_{\mathbb{P}^1} B_2$ is constructed. A priori, the dP_9 surfaces $B_1 \rightarrow \mathbb{P}^1$ and $B_2 \rightarrow \mathbb{P}^1$ are elliptically fibered over two different \mathbb{P}^1 . In making the fiber product, one has to identify the \mathbb{P}^1 bases. But one can always choose coordinates to fix 3 points on the sphere! Hence we can always pick a complex structure of X such that

$$q_1 = p_1, \quad q_2 = p_2, \quad s_1 = r_1. \quad (5.49)$$

For this particular complex structure,

$$\text{Hsupp}(L^{-4}) \cap \text{Hsupp}(W^\vee \otimes \det^{-1} W) = \{p_1, p_2, r_1\}, \quad (5.50)$$

and therefore

$$\text{Ext}^1(V_3, V_1) = \mathbb{C}^3 \neq 0. \quad (5.51)$$

Now we will show that with the identification eq. (5.49) and assuming that p_1 and p_2 appear in $\text{Hsupp}(\wedge^2 W)$ with multiplicity two (see Section 6 for details), we can make the number of the $SO(10)$ fundamentals

$$N_f = h^1(X, \wedge^2 V) = 8. \quad (5.52)$$

This number satisfies the inequality eq. (2.11) for $N_c = 10$. In other words, we will prove that in the moduli space of the complex structure and vector bundle there is at least one locus where exactly 8 fundamentals become light. The spectral cover remains irreducible and the vector bundle remains smooth and stable along this locus. All other matter fields are massive and integrated out. Moving slight away from this locus gives light masses to these eight fundamental multiplets. This is exactly what is need to satisfy the criteria stated in Section 2. To move away from this locus, for example, means to slightly separate p_1 from q_1 and p_2 from q_2 . This is controlled by complex structure and/or vector bundle moduli. In the next subsection, we will argue that the 8 fundamentals of interest receive a superpotential of the form eq. (3.1).

To compute cohomology of $\wedge^2 V$ we have to relate it to cohomology of V_1 and V_3 . From the maps in eq. (5.12) we can construct two exact sequences

$$\begin{aligned} 0 \longrightarrow \wedge^2 V_1 \longrightarrow \wedge^2 V \longrightarrow Q_1 \longrightarrow 0, \\ 0 \longrightarrow Q_2 \longrightarrow \wedge^2 V \longrightarrow \wedge^2 V_3 \longrightarrow 0 \end{aligned} \quad (5.53)$$

for some cokernel Q_1 and kernel Q_2 . These two exact sequences fit together into the commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^2 V_1 & \longrightarrow & Q_2 & \longrightarrow & V_1 \otimes V_3 \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & \wedge^2 V_1 & \longrightarrow & \wedge^2 V & \longrightarrow & Q_1 \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & \wedge^2 V_3 & \xlongequal{\quad} & \wedge^2 V_3 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array} \tag{5.54}$$

with exact rows and columns. In our case $\wedge^2 V_1 = 0$ is the rank 0 vector bundle, since V_1 is a line bundle. Therefore the commutative diagram simplifies to the short exact sequence

$$0 \longrightarrow V_1 \otimes V_3 \longrightarrow \wedge^2 V \longrightarrow \wedge^2 V_3 \longrightarrow 0. \tag{5.55}$$

for $\wedge^2 V$. Using the definitions eqns. (5.21) and (5.21), the outer terms are

$$\begin{aligned}
\wedge^2 V_3 &= \pi_1^*(L^2) \otimes \pi_2^*(\wedge^2 W), \\
V_1 \otimes V_3 &= \pi_1^*(L^{-2}) \otimes \pi_2^*(W \otimes \det^{-1} W) = \pi_1^*(L^{-2}) \otimes \pi_2^*(\wedge^2 W^\vee).
\end{aligned} \tag{5.56}$$

If we abbreviate

$$\mathcal{F} = [\pi_1^*(L^2) \otimes \pi_2^*(\wedge^2 W)]^\vee, \tag{5.57}$$

then the sequence eq. (5.55) can be written as

$$0 \longrightarrow \mathcal{F} \longrightarrow \wedge^2 V \longrightarrow \mathcal{F}^\vee \longrightarrow 0. \tag{5.58}$$

Now we can use the long exact sequence of cohomology to relate the cohomology groups of $\wedge^2 V$ to the cohomology groups of \mathcal{F} . Since $\wedge^2 V$ is self-dual, Serre duality tells us that $h^1(X, \wedge^2 V) = h^2(X, \wedge^2 V)$. Hence we can concentrate on the part of the sequence involving $H^2(X, \wedge^2 V)$, which reads

$$\cdots \xrightarrow{\delta} H^2(X, \mathcal{F}) \longrightarrow H^2(X, \wedge^2 V) \longrightarrow H^2(X, \mathcal{F}^\vee) \longrightarrow H^3(X, \mathcal{F}) = 0, \tag{5.59}$$

where

$$\delta : H^1(X, \mathcal{F}^\vee) \longrightarrow H^2(X, \mathcal{F}) \tag{5.60}$$

is a coboundary map, which we can think of as a matrix with entries depending on vector bundle moduli. It is determined by the chosen extension class

$$[\epsilon] \in \text{Ext}^1(V_3, V_1) = H^1(X, V_1 \otimes V_3^\vee) = H^1\left(X, \pi_1^*(L^{-4}) \otimes \pi_2^*(W^\vee \otimes \det^{-1} W)\right). \tag{5.61}$$

The coboundary map eq. (5.60) is simply multiplication by ϵ followed by a suitable contraction of vector bundle indices. The vector bundle extension, that is the cohomology of the vector bundle $V_1 \otimes V_3^\vee$, is supported at

$$\text{Hsupp}(L^{-4}) \cap \text{Hsupp}(W^\vee \otimes \det^{-1} W) = \{p_1, p_2, r_1\}, \quad (5.62)$$

whereas the cohomology of \mathcal{F}^\vee , \mathcal{F} is supported at

$$\text{Hsupp}(L^2) \cap \text{Hsupp}(\wedge^2 W) = \{p_1, p_2\}. \quad (5.63)$$

Observe that the support of the extension class contains an additional point over the support of the cohomology of \mathcal{F}^\vee , \mathcal{F} . Hence, we can choose the extension class $[\epsilon]$ to be localized at this additional point r_1 , and we are going to do so in the following. In that case the coboundary map δ is automatically zero, and the sequence eq. (5.59) becomes

$$0 \longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(X, \wedge^2 V) \longrightarrow H^2(X, \mathcal{F}^\vee) \longrightarrow 0. \quad (5.64)$$

Therefore,

$$h^2(X, \wedge^2 V) = h^2(X, \mathcal{F}) + h^2(X, \mathcal{F}^\vee). \quad (5.65)$$

The cohomology group $H^2(X, \mathcal{F}^\vee)$ is straightforward to calculate using the Leray spectral sequence, see also eq. (5.33). The answer is

$$H^2(X, \mathcal{F}^\vee) = H^0(\mathbb{P}^1, R^1\beta_{1*}(L^2) \otimes R^1\beta_{2*}(\wedge^2 W)). \quad (5.66)$$

We only have to be careful with the multiplicity of points in $\text{Hsupp}(\wedge^2 W)$. As discussed before, the push-down terms are skyscraper sheaves supported at the points

$$\begin{aligned} R^1\beta_{1*}(L^2) &= \mathcal{O}_{q_1} \oplus \mathcal{O}_{q_2} \oplus \mathcal{O}_{q_3}, \\ R^1\beta_{2*}(\wedge^2 W) &= 2\mathcal{O}_{p_1} \oplus 2\mathcal{O}_{p_2} \oplus \mathcal{O}_{p_3}, \end{aligned} \quad (5.67)$$

where \mathcal{O}_p denotes the “skyscraper” sheaf which is a one-dimensional vector space at p at zero everywhere else. Recalling our identifications eq. (5.49), we obtain⁵

$$R^1\beta_{1*}(L^2) \otimes R^1\beta_{2*}(\wedge^2 W) = 2\mathcal{O}_{p_1} \oplus 2\mathcal{O}_{p_2}. \quad (5.68)$$

Then using eq. (5.66) it follows that

$$h^2(X, \mathcal{F}^\vee) = h^0(\mathbb{P}^1, 2\mathcal{O}_{p_1} \oplus 2\mathcal{O}_{p_2}) = 4. \quad (5.69)$$

In exactly the same way one arrives at

$$h^2(X, \mathcal{F}) = 4, \quad (5.70)$$

⁵Recall that $\mathcal{O}_p \otimes \mathcal{O}_p = \mathcal{O}_p$ whereas $\mathcal{O}_p \otimes \mathcal{O}_q = 0$ for $p \neq q$.

as well. Therefore, we find from eq. (5.65) that

$$N_f = h^2(X, \wedge^2 V) = h^1(X, \wedge^2 V) = 8. \quad (5.71)$$

Thus, our model indeed has $N_f = 8$ massless fundamental multiplets, satisfying the inequality eq. (2.11) for $N_c = 10$. As discussed earlier in this subsection we can give them small masses. In the next subsection we will show that they have a superpotential of the form eq. (3.1).

5.6 Different Numbers of Flavors

In the following, we will always stick to the $N_f = 8$ case in order to make everything as explicit as possible. However, one can easily construct similar bundles yielding different values for N_f . Let us explore these possibilities.

- One simple change would be to deform one of the ordinary double points such that

$$\text{Hsupp}(\wedge^2 W') = \{2p_1, p_2, p'_2, p_3\} \quad (5.72)$$

for the new spectral curve C' . This is achieved by modifying the moduli of W . In terms of the equations for the curve to be discussed in Section 6, this amounts to allowing the cubic F_2 , eq. (6.13), to be arbitrary. Following exactly the same steps as in Subsection 5.5, this yields $N_f = 6$.

- Similarly, by modifying the vector bundle moduli, it is easy to obtain any even N_f less than six. For completeness, let us discuss this case even though it does not satisfy (2.11). Consider the regime in the moduli space where

$$\text{Hsupp}(\wedge^2 W') = \{p_1, p'_1, p_2, p'_2, p_3\} \quad (5.73)$$

A calculation analogous to the one performed in the previous subsection yields $N_f = 4$. Now assuming that $\text{Hsupp}(\wedge^2 W)$ does not have any double points, let us move in the moduli space of complex structures so that the point $q_2 \in \text{Hsupp}(L^2)$ gets separated from $p_2 \in \text{Hsupp}(\wedge^2 W)$ and is not identified with any other point of $\text{Hsupp}(\wedge^2 W)$. Then the analysis of the previous subsection yields $N_f = 2$. Note that this separation also changes the possible non-trivial extensions. From eqns. (5.45) and (5.48) it follows that

$$\text{Ext}^1(V_3, V_1) = \mathbb{C}^2. \quad (5.74)$$

Separating further q_1 and p_1 makes the supports $\text{Hsupp}(L^2)$ and $\text{Hsupp}(\wedge^2 W)$ completely disjoint. This yields $N_f = 0$. From eqns. (5.45) and (5.48) it follows that in this case

$$\text{Ext}^1(V_3, V_1) = \mathbb{C}. \quad (5.75)$$

- Another easy modification is to take a spectral curve C'' with one ordinary double points and one ordinary triple point. In terms of the intersection points $W''|_{f_1}$ this means (see eq. (6.9)) that

$$S_1'' = \Xi S_1'' = T_1'' . \quad (5.76)$$

To have enough parameters to adjust the spectral curve, one needs $C'' \in \Gamma \mathcal{O}_{B_2}(\ell + 2f)$, which we are free to choose. The only change to the matter spectrum of the resulting vector bundle is that now $N_f = 10$. Similarly, by considering the spectral cover of W to be in $\Gamma \mathcal{O}_{B_2}(\ell + nf)$ for greater values of n , it is possible to obtain greater values of N_f .

- The fact that N_f was always even so far is an artifact of the rank 3 bundle W . This can be relaxed, for example, by constructing rank 5 bundles as extension of a line bundle and a rank 4 bundle W''' . The same trick of aligning points on the base \mathbb{P}^1 can then be used to adjust the matter spectrum. In this way, one can find odd N_f lying in the range eq. (2.11).

To conclude, we have shown that we can obtain the spectrum with $N_f = 0, \dots, 10, \dots$. Thus, in particular, we have shown that we can find any N_f in the range (2.11).

5.7 The Superpotential

Let us again consider the exact sequence eq. (5.59). Now let us pick a generic extension ϵ instead of one supported only at r_1 . In that case the bundle extension is supported at the points p_1, p_2 which support the cohomology of \mathcal{F} . Then, generically, the coboundary map δ becomes an isomorphism and the exact sequence eq. (5.59) becomes

$$0 \longrightarrow H^2(X, \wedge^2 V) \xrightarrow{\sim} H^2(X, \mathcal{F}^\vee) \longrightarrow 0 , \quad (5.77)$$

resulting in

$$N_f = h^1(X, \wedge^2 V) = h^2(X, \wedge^2 V) = h^2(X, \mathcal{F}^\vee) = 4 . \quad (5.78)$$

In other words, turning on the vector bundle moduli parametrizing $\text{Ext}^1(V_3, V_1)$ we can remove half of $H^2(X, \wedge^2 V)$. Similarly, turning on the anti-extension moduli coming from $\text{Ext}^1(V_1, V_3)$ we can remove the other half of $H^2(X, \wedge^2 V)$. This means that we have a superpotential that is quadratic in the elements of $H^2(X, \wedge^2 V)$, giving mass to all fields at a generic point in the moduli space.

Let us finish by giving a general explanation why coboundary maps correspond to a cubic superpotential of the form in eq. (3.1). Let the vector bundle U be the extension of U_1 and U_2 ,

$$0 \longrightarrow U_1 \longrightarrow U \longrightarrow U_2 \longrightarrow 0 . \quad (5.79)$$

In the long exact sequence for the cohomology there is a coboundary map

$$\delta : H^1(X, U_2) \rightarrow H^2(X, U_1) = H^1(X, U_1^\vee)^\vee. \quad (5.80)$$

The map δ is a multiplication by a matrix ϵ of differential forms parametrized by the vector bundle moduli. It is an element of the extension group $[\epsilon] \in \text{Ext}^1(U_2, U_1) = H^1(X, U_1 \otimes U_2^\vee)$. Eq. (5.80) says that the tensor product $H^1(X, U_2) \otimes \text{Ext}^1(U_2, U_1)$ defines an element in the dual space $H^1(X, U_1^\vee)^\vee$. Elements of $H^1(X, U_1^\vee)^\vee$ can naturally be paired up with elements of $H^1(X, U_1^\vee)$ to obtain a complex number. Thus, we can rewrite this map as

$$H^1(X, U_2) \otimes \text{Ext}^1(U_2, U_1) \otimes H^1(X, U_1^\vee) \rightarrow \mathbb{C}. \quad (5.81)$$

Looking at the long exact sequence in cohomology, elements of $H^1(X, U_2)$ label a quotient of $H^1(X, U)$. Similarly, $H^1(X, U_1^\vee)$ is a quotient of $H^1(X, U^\vee)$. In our case $\wedge^2 V$ is real so both are some quotient of the same space $H^1(X, \wedge^2 V)$. The corresponding four-dimensional fields Q are in a real representation of the low-energy gauge group. Finally, elements of $\text{Ext}^1(U_2, U_1)$ are part of the vector bundle moduli [58, 59]. Denote them as ϕ . Then the map eq. (5.81) is the algebraic version of the superpotential eq. (3.1).

6 The Geometry of the Spectral Cover

6.1 Requirements

In this section, we will give a detailed explanations of why the support of $\wedge^2 W$ can of the form eq. (5.47). Recall that, as in eq. (4.5), the restriction of W to a generic fiber f is

$$W|_f = \mathcal{O}_f(C_1 - o) \oplus \mathcal{O}_f(C_2 - o) \oplus \mathcal{O}_f(C_3 - o), \quad (6.1)$$

where the points C_1, C_2, C_3 , and o on f are intersection points with the spectral curve C_W and the zero section σ ,

$$\{C_1, C_2, C_3\} = C_W \cdot f, \quad o = \sigma \cdot f. \quad (6.2)$$

Tensor operations commute with the restriction, so we can simply write down⁶

$$\begin{aligned} \wedge^2 W|_f &= \mathcal{O}_f(C_1 \boxplus C_2 - o) \oplus \mathcal{O}_f(C_1 \boxplus C_3 - o) \oplus \mathcal{O}_f(C_2 \boxplus C_3 - o), \\ \wedge^3 W|_f &= \det W|_f = \mathcal{O}_f(C_1 \boxplus C_2 \boxplus C_3 - o). \end{aligned} \quad (6.3)$$

Now we want a special spectral cover such that the cohomology support $\text{Hsupp}(\wedge^2 W)$ has a pair of points with multiplicity 2. In other words, on two special fibers

$$f_1 = \beta_2^{-1}(p_1), \quad f_2 = \beta_2^{-1}(p_2) \quad (6.4)$$

⁶The group law \boxplus on the elliptic curve f satisfies $\mathcal{O}_f(p - o) \otimes \mathcal{O}_f(q - o) = \mathcal{O}_f((p \boxplus q) - o)$.

we want (labeling the origin $o_i = \sigma \cdot f_i$)

$$\begin{aligned}\wedge^2 W|_{f_1} &= \mathcal{O}_{f_1} \oplus \mathcal{O}_{f_1} \oplus \mathcal{O}_{f_1}(a_1 - o_1), \\ \wedge^2 W|_{f_2} &= \mathcal{O}_{f_2} \oplus \mathcal{O}_{f_2} \oplus \mathcal{O}_{f_2}(a_2 - o_2).\end{aligned}\tag{6.5}$$

for some points $a_i \in f_i - \{o_i\}$, $i = 1, 2$. In terms of the spectral curve C_W , this means that we want

$$C_W \cdot f_i = \{2S_i, T_i\}\tag{6.6}$$

satisfying

$$S_i \boxplus T_i = o_i, \quad S_i \boxplus S_i = a_i \neq o_i.\tag{6.7}$$

Note that there are two ways to achieve intersections with multiplicities as in eq. (6.6). Deforming the fiber away from f_i , the intersection points with C_W have to split up into 3 distinct points. This triple can have a monodromy as one moves around f_i , or it can have no monodromy. In the first case the spectral curve C_W has a branch point, while in the second case the spectral curve has an ordinary double point. The corresponding spectral curve for $\wedge^2 W$ has a worse singularity in the first case, and again an ordinary double point in the second case. Now an ordinary double point is simply a transverse intersection of two different sheets of C_W . While technically called a singularity, it does not change anything for the spectral cover construction. Hence, we will demand that the points S_1 and S_2 are ordinary double points of C_W . Such a spectral curve would yield

$$W|_{f_i} = \mathcal{O}_{f_i}(S_i - o_i) \oplus \mathcal{O}_{f_i}(S_i - o_i) \oplus \mathcal{O}_{f_i}(T_i - o_i),\tag{6.8a}$$

$$W^\vee|_{f_i} = \mathcal{O}_{f_i}(\boxminus S_i - o_i) \oplus \mathcal{O}_{f_i}(\boxminus S_i - o_i) \oplus \mathcal{O}_{f_i}(\boxminus T_i - o_i),\tag{6.8b}$$

$$\wedge^2 W|_{f_i} = \mathcal{O}_{f_i} \oplus \mathcal{O}_{f_i} \oplus \mathcal{O}_{f_i}(S_i \boxplus S_i - o_i),\tag{6.8c}$$

$$\det W|_{f_i} = \mathcal{O}_{f_i}(S_i \boxplus S_i \boxplus T_i - o_i) = \mathcal{O}_{f_i}(S_i - o_i),\tag{6.8d}$$

$$\left(W^\vee \otimes \det^{-1} W\right)|_{f_i} = \mathcal{O}_{f_i}(\boxminus S_i \boxminus S_i - o_i) \oplus \mathcal{O}_{f_i}(\boxminus S_i \boxminus S_i - o_i) \oplus \mathcal{O}_{f_i},\tag{6.8e}$$

as desired. Note that the last equation, eq. (6.8e), tells us that the cohomology of $W^\vee \otimes \det^{-1} W$ is also supported at p_1 and p_2 , which we announced previously in eq. (5.48).

To summarize, we require that our spectral curve satisfies

- $C_W \in \Gamma \mathcal{O}_{B_2}(\ell + f)$, see eq. (5.15).
- C_W has 2 ordinary double points S_1 and S_2 in two different fibers f_1 and f_2 , which we take to be non-degenerate elliptic curves for simplicity.
- Then there are two more points T_1, T_2 satisfying

$$f_i \cdot C_W = \{2S_i, T_i\}.\tag{6.9}$$

With respect to the group law on the elliptic curves, we require that

$$S_i \boxplus T_i = \sigma \cdot f_i \quad (6.10)$$

- The double points do not coincide with the origin, that is

$$S_i \neq \sigma \cdot f_i \quad \Leftrightarrow \quad T_i \neq \sigma \cdot f_i \quad \Leftrightarrow \quad \det W|_{f_i} \neq \mathcal{O}_{f_i}. \quad (6.11)$$

6.2 The Pencil of Cubics

So far, we assumed the existence of a suitable spectral curve C_W in order to construct our vector bundle. Given that the surface B_2 has 10 and the curve C_W has 5 parameters, it is very plausible that some choice of dP₉ surface and curve actually satisfies the requirements laid out in Subsection 6.1. The purpose of this section is to write an explicit spectral curve and show that it satisfies all requirements. This will establish the existence of curve C_W and, hence, of our vector bundle.

First, we have to specify the actual dP₉ surface B_2 . We define it as a ‘‘Pencil of Cubics’’, that is, a bi-degree $(3, 1)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$. In the following, we are going to use coordinates $[x : y : z]$ for the coordinates on \mathbb{P}^2 and $[u : v]$ for \mathbb{P}^1 . Define the two cubics

$$\begin{aligned} F_1(x, y, z) &= (x - y)(x - z)(x + z) + z(x^2 + y^2 - z^2 - 2yx - 4zx + 5yz) \\ F_2(x, y, z) &= (x - z)(x - y)(x + y) + y(x^2 + z^2 - y^2 - 2zx - 4yx + 5yz) \\ &= F_1(x, z, y), \end{aligned} \quad (6.12)$$

then

$$P(x, y, z; u, v) = u F_1(x, y, z) + v F_2(x, y, z) \quad (6.13)$$

is the desired equation. We define

$$B_2 = \{P = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1. \quad (6.14)$$

The elliptic fibration $\beta_2 : B_2 \rightarrow \mathbb{P}^1$ is just the projection on the second factor, and we write

$$f_{[u_0:v_0]} = \beta_2^{-1}([u_0 : v_0]) \quad (6.15)$$

for the fiber over $[u_0 : v_0] \in \mathbb{P}^1$. The discriminant locus of the elliptic fibration is

$$\begin{aligned} \Delta(P) = \frac{25}{16} & \left(131 v^{10} + 5774 uv^9 - 94185 u^2 v^8 \right. \\ & - 2553672 u^3 v^7 + 26073510 u^4 v^6 - 49632012 u^5 v^5 + 26073510 u^6 v^4 \\ & \left. - 2553672 u^7 v^3 - 94185 u^8 v^2 + 5774 u^9 v + 131 u^{10} \right) (u + v)^2 \end{aligned} \quad (6.16)$$

We observe that B_2 is a smooth surface, but of course some fibers of the elliptic fibration are degenerate. More precisely, B_2 has $10I_1$ and $1I_2$ singular Kodaira fibers, none of which lie over the two points $[u : v] = [1 : 0], [0 : 1]$.

Note that the point $[2 : 1 : 1] \in \mathbb{P}^2$ is a basepoint of the pencil of cubics (of multiplicity 1). That is,

$$F_1(2, 1, 1) = 0 = F_2(2, 1, 1), \quad -10 = \frac{\partial F_1}{\partial y} \Big|_{(2,1,1)} \neq \frac{\partial F_2}{\partial y} \Big|_{(2,1,1)} = 0. \quad (6.17)$$

Such a basepoint defines a section of the elliptic fibration, which we declare to be the zero section

$$e_9 = \left\{ ([2 : 1 : 1], [u : v]) \mid [u : v] \in \mathbb{P}^1 \right\} \subset B_2 \quad (6.18)$$

6.3 The Spectral Curve

Having fixed the dP_9 surface B_2 , we are now going to pick a curve C_W on it. For that, we define the equation

$$Q(x, y, z; u, v) = vz - uy \quad (6.19)$$

on $B_2 \subset \mathbb{P}^2 \times \mathbb{P}^1$. Its zero locus will be the curve

$$C_W = \{Q = 0\} \subset B_2. \quad (6.20)$$

Clearly, C is a 3-section of the elliptic fibration since its intersection with the fiber over $[u_0 : v_0] \in \mathbb{P}^1$ is given by the cubic equation

$$C_W|_{f_{[u_0:v_0]}} = \left\{ P(x, y, z; u_0, v_0) = 0 = Q(x, y, z; u_0, v_0) \right\}. \quad (6.21)$$

Note that a degree $(1, 0)$ equation is, by definition, the hyperplane section of \mathbb{P}^2 , which is the homology class

$$[\{x = 0\}] = [\{y = 0\}] = [\{z = 0\}] = \ell \in H_2(B_2, \mathbb{Z}). \quad (6.22)$$

Similarly, a degree $(0, 1)$ equation cuts out one elliptic fiber of B_2 ,

$$[\{u = 0\}] = [\{v = 0\}] = f \in H_2(B_2, \mathbb{Z}). \quad (6.23)$$

Therefore

$$C_W \in \Gamma\mathcal{O}_{B_2}(\ell + f) \quad \Rightarrow \quad [C_W] = \ell + f \in H_2(B_2). \quad (6.24)$$

Computing the monodromies around branch points of C_W , we find that it is an irreducible curve.

The curve C_W is singular since having two “ordinary double points” was part of the requirements. These two points are

$$\begin{aligned} S_1 &= ([0 : 0 : 1], [1 : 0]) \in f_{[1:0]}, \\ S_2 &= ([0 : 1 : 0], [0 : 1]) \in f_{[0:1]}. \end{aligned} \quad (6.25)$$

Since each fiber contains 3 points of C_W (counted with multiplicity), there is another point in $f_{[1:0]}$ and $f_{[0:1]}$, respectively. They are smooth points of C_W , and we label them

$$\begin{aligned} T_1 &= ([1 : 0 : 1], [1 : 0]) \in f_{[1:0]}, \\ T_2 &= ([1 : 1 : 0], [0 : 1]) \in f_{[0:1]}. \end{aligned} \quad (6.26)$$

Apart from S_1 and S_2 , there are no other singularities. As a 3-sheeted cover of the base \mathbb{P}^1 there are 6 branch points in other fibers, this is depicted in Figure 1.

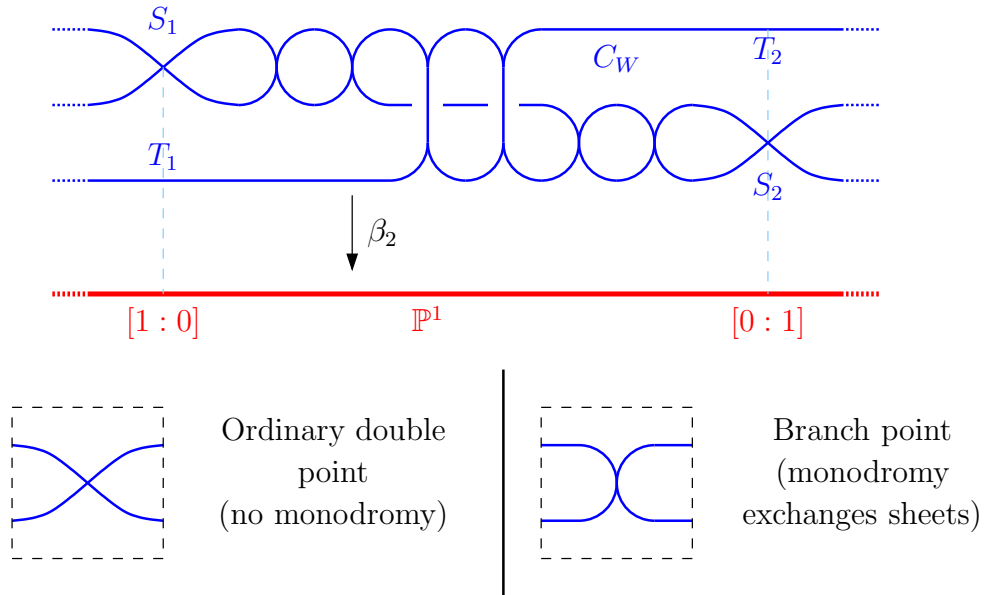


Figure 1: The 3-section C_W .

It remains to show that

$$S_1 \boxplus T_1 = 0, \quad S_2 \boxplus T_2 = 0 \quad (6.27)$$

in the group law on the respective fibers, then the curve C_W satisfies all requirements. Note that everything so far was by construction symmetric under the exchange

$$([x : y : z], [u : v]) \leftrightarrow ([x : z : y], [v : u]) \quad (6.28)$$

| \mathbb{P}^2 | S_1 | T_2 | $e_9 \cap f_{[0:1]}$ |
|---------------------------|---------------|----------------|----------------------|
| $[x : y : z]$ coordinates | $[0 : 1 : 0]$ | $[1 : 1 : 0]$ | $[2 : 1 : 1]$ |
| $[X : Y : Z]$ coordinates | $[1 : 5 : 1]$ | $[1 : -5 : 1]$ | $[0 : 1 : 0]$ |

Table 1: Coordinate transformation to Weierstrass form

Because of this symmetry, it suffices to show that $S_2 \boxplus T_2 = 0$. The elliptic curve $f_{[0:1]} \subset \mathbb{P}^2$ is given by the cubic

$$P(x, y, z; 0, 1) = x^3 - z^3 - 5xz^2 - x^2y + 6yz^2 + x^2z + y^2z - 2xyz \quad (6.29)$$

with origin

$$e_9 \cap f_{[0:1]} = [2 : 1 : 1] \in \mathbb{P}^2 \quad (6.30)$$

To bring the cubic into Weierstrass form we have to do a birational transformation of the \mathbb{P}^2 . Specifically, we choose new projective coordinates $[X : Y : Z]$ via

$$\begin{aligned} x &= 2(X - Z)(2X - 7Z), \\ y &= 2X^2 - 34XZ^2 + 57Z^3 + 5YZ^2, \\ z &= 2(X - Z)^2, \end{aligned} \quad (6.31)$$

which maps the chosen origin to⁷ $[0 : 1 : 0]$ in the new coordinates. Substituting into eq. (6.29) we find

$$P(X, Y, Z; 0, 1) = -50Z(X - Z)^2(-Y^2Z + 4X^3 - 52XZ^2 + 73Z^3) \quad (6.32)$$

Hence, the Weierstrass form of our elliptic curve is

$$Y^2Z = 4X^3 - 52XZ^2 + 73Z^3 \quad (6.33)$$

The coordinates of the points S_2 and T_2 turn out to be at the locus where the birational transformation is not defined, but one can still find their values by continuity. The new coordinates of the relevant points are listed in Table 1. Recall that the inverse in the group law of the cubic has a particularly nice form for a cubic in Weierstrass form, it is

$$\boxminus[X : Y : Z] = [X : -Y : Z]. \quad (6.34)$$

Hence, we immediately realize that

$$\boxminus S_2 = T_2 \quad \Leftrightarrow \quad S_2 \boxplus T_2 = 0, \quad (6.35)$$

as required.

⁷ $[0 : 1 : 0]$ is the origin in the Weierstrass form of a cubic.

7 Conclusion and Further Directions

In this paper, we addressed the question of realizing dynamical SUSY breaking [14] in heterotic model building. We discussed how quadratic superpotentials for matter fields arise in heterotic compactifications. The mass of these fields depends on the complex structure and vector bundle moduli. Thus, by moving in the moduli space, we can make some of the matter fields either very light or very heavy. From an algebraic geometry viewpoint, this means that the dimension of various cohomology groups associated to the number of matter particles jumps as we move in the moduli space. We present a stable, holomorphic hidden sector bundle satisfying the criteria for dynamical SUSY breaking. The main example studied in this paper is $SO(10)$ SQCD with $N_f = 8$ fundamental fields. All other matter fields are heavy and integrated out. We give a detailed analysis showing that there is a locus in the moduli space where exactly eight fundamentals become massless whereas all other matter is massive. Moving slightly away from this locus is equivalent to generate the superpotential eq. (3.1). Hidden sectors for different values of N_f can be constructed analogously. This is discussed in subsection 5.6. In particular, it is shown that it is possible to obtain $SO(10)$ SQCD with any number of fundamental fields in the range (2.11)

Let us briefly discuss various generalizations of these results. One natural direction is to construct the hidden sector breaking supersymmetry in realistic standard model compactifications [7, 8, 9, 10, 11, 12]. That is, in addition to the sector whose particle spectrum is that of a supersymmetric standard model, to put a hidden sector vector bundle (presumably one without Wilson lines) that will lead to one of the theories studied in [14]. Another direction would be to understand the F-theory dual [60, 54] of a model studied in this paper. The F-theory dual space is a Calabi-Yau fourfold Y and the matter is supposed to be localized on intersecting $D7$ -branes wrapping four-cycles of Y . The moduli of the heterotic vector bundle will be mapped to certain geometric moduli of Y . Thus, giving mass to the fundamentals by means of the superpotential eq. (3.1) on the F-theory side will have a geometric interpretation as brane separation. Having this interpretation, it might be easier to understand the location of the loci where the right number of the fundamental multiplets receive a small mass. Another possible advantage of it is that it could be easier to understand under what conditions the moduli controlling the masses of the fundamentals can be stabilized in a regime of interest. Unfortunately, it is not yet known how the heterotic/F-theory duality map acts on the spectrum.

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A The Support of L^k

Let $L = \mathcal{O}_B(s_1 - s_2)$ be a line bundle on a dP_9 surface B , and s_1 and s_2 be two sections that do not intersect. For example, one can use $s_1 = e_1$ and $s_2 = e_9$ where e_1 and e_9 are two out of the nine exceptional divisors from the blow-up. Thus both e_1 and e_9 are isomorphic to \mathbb{P}^1 . We will denote by β the projection of B to the base \mathbb{P}^1 . Via the Leray spectral sequence, the cohomology of L is determined by the cohomology (on \mathbb{P}^1) with coefficients in either β_*L or $R^1\beta_*L$. In particular, as the base is 1-dimensional one obtains

$$\begin{aligned} H^0(B, L) &= H^0(\mathbb{P}^1, \beta_*L), \\ H^1(B, L) &= H^0(\mathbb{P}^1, R^1\beta_*L) \oplus H^1(\mathbb{P}^1, \beta_*L), \\ H^2(B, L) &= H^1(\mathbb{P}^1, R^1\beta_*L). \end{aligned} \tag{A.1}$$

Since s_1 and s_2 do not intersect, the restriction of L to any fiber gives a non-trivial line bundle of degree zero on elliptic curve, see Section 4. Such a line bundle (on a fiber) has no cohomology, that is, all cohomology groups vanish. Therefore, both β_*L or $R^1\beta_*L$ are the zero sheaf

$$\beta_*L = R^1\beta_*L = 0. \tag{A.2}$$

This means that in turn all cohomology groups (on B) of L vanish,

$$H^*(B, L) = 0. \tag{A.3}$$

Let us now consider $L^2 = \mathcal{O}_B(2s_1 - 2s_2)$. As we will see below, for this line bundle β_*L^2 is still zero. However, $R^1\beta_*L^2$ is not zero. Instead, it is a torsion sheaf. According to the Leray spectral sequence,

$$H^1(B, L^2) = H^0(\mathbb{P}^1, R^1\beta_*L^2). \tag{A.4}$$

The right hand side is just the number of points at which $R^1\beta_*L^2$ is supported. Let us calculate this number. To do so we will be using the following standard exact sequence [47]. Let D be any effective divisor on a manifold Z , then the sequence

$$0 \longrightarrow \mathcal{O}_Z(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_D \longrightarrow 0. \tag{A.5}$$

is exact. The first map $\mathcal{O}_Z(-D) \rightarrow \mathcal{O}_Z$ is multiplication by a global section of $\mathcal{O}_Z(D)$ which vanishes exactly at D . The second map $\mathcal{O}_Z \rightarrow \mathcal{O}_D$ is simply the restriction to $D \subset Z$.

Let us apply it to the case when $Z \simeq \mathbb{P}^1$ and $D \simeq \{\text{pt.}\}$. Then we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_p \longrightarrow 0. \quad (\text{A.6})$$

The cokernel \mathcal{O}_p is the skyscraper sheaf supported at a point p . This sequence can easily be generalized for the case of n points, and the cokernel of the inclusion map

$$i : \mathcal{O}_{\mathbb{P}^1}(-n) \rightarrow \mathcal{O}_{\mathbb{P}^1} \quad (\text{A.7})$$

is a skyscraper sheaf supported at n points. The detailed location of these points depends on the map i which is multiplication by a global section of $\mathcal{O}(n)$. Any such section has n zeroes which are the support of the cokernel of i . Conversely, for any n points there is such a (unique up to an overall constant) section vanishing at the n points.

Now let us apply the short exact sequence eq. (A.4) to the bundle L^2 . First we consider the sequence

$$0 \longrightarrow \mathcal{O}_B(s_1 - s_2) \longrightarrow \mathcal{O}_B(2s_1 - s_2) \longrightarrow \mathcal{O}_{s_1}(2s_1 \cdot s_1) \longrightarrow 0. \quad (\text{A.8})$$

The bundle $\mathcal{O}_{s_1}(2s_1 \cdot s_1)$ is a bundle on $s_1 \sim \mathbb{P}^1$ of degree $-2 = 2s_1^2$. This short exact sequence on B leads to a long exact sequence on \mathbb{P}^1 of the direct images

$$0 \longrightarrow \beta_* \mathcal{O}_B(s_1 - s_2) \longrightarrow \beta_* \mathcal{O}_B(2s_1 - s_2) \rightarrow \mathcal{O}(-2) \longrightarrow R^1 \beta_* \mathcal{O}_B(s_1 - s_2) \longrightarrow \dots \quad (\text{A.9})$$

We have shown above that $\beta_* \mathcal{O}_B(s_1 - s_2) = R^1 \beta_* \mathcal{O}_B(s_1 - s_2) = 0$. Therefore,

$$\begin{aligned} \beta_* \mathcal{O}_B(2s_1 - s_2) &= \mathcal{O}_{\mathbb{P}^1}(-2), \\ R^1 \beta_* \mathcal{O}_B(2s_1 - s_2) &= 0 \end{aligned} \quad (\text{A.10})$$

Finally, consider the sequence

$$0 \longrightarrow \mathcal{O}_B(2s_1 - 2s_2) \longrightarrow \mathcal{O}_B(2s_1 - s_2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0, \quad (\text{A.11})$$

where we used the intersection numbers eq. (5.5) already. Inserting eq. (A.10), we find direct images

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow R^1 \beta_*(L^2) \longrightarrow 0. \quad (\text{A.12})$$

From the above discussion it follows that $R^1 \beta_*(L^2)$ is the skyscraper sheaf supported at 3 points. In Section 5, we denoted these points by q_1, q_2, q_3 . Similarly, one can show that the sheaf $R^1 \beta_*(L^3)$ is supported at 8 points and the sheaf $R^1 \beta_*(L^4)$ is supported at 15 points. It is not hard to show that these 15 points contain the points q_1, q_2 , and q_3 each with multiplicity one.

B Derived Tensor Products

Consider the case of a bundle on $X = B_1 \times_{\mathbb{P}^1} B_2$ constructed as

$$U = \pi_1^*(U_1) \otimes \pi_2^*(U_2), \quad (\text{B.1})$$

as we are using throughout this paper. Moreover, let the bundles U_i on B_i be semistable of fiber-degree zero. For ease of presentation, we assume that their direct image contains only a single skyscraper sheaf, that is,

$$\beta_{1*}(U_1) = 0, \quad R^1\beta_{1*}(U_1) = \mathcal{O}_p, \quad (\text{B.2})$$

$$\beta_{2*}(U_2) = 0, \quad R^1\beta_{2*}(U_2) = \mathcal{O}_q \quad (\text{B.3})$$

for two points $p, q \in \mathbb{P}^1$. To compute the cohomology we can apply the Leray spectral sequence, either pushing down to B_1 or to B_2 ,

$$\begin{aligned} H^i(X, U) &= \bigoplus_{n+m=i} H^n(U_1 \otimes \beta_1^* \circ R^m \beta_{2*}(U_2)) = \\ &= \bigoplus_{n+m=i} H^n(\beta_2^* \circ R^m \beta_{1*}(U_1) \otimes U_2) \end{aligned} \quad (\text{B.4})$$

However, a problem arises when one attempts to push either term further down to \mathbb{P}^1 . Because the $R^1\beta_{i*}(U_i)$ is not a vector bundle we cannot simply apply the projection formula, and

$$\begin{aligned} R^n\beta_{1*}(U_1) \otimes R^m\beta_{2*}(U_2) &\neq R^n\beta_{1*}\left(U_1 \otimes \left[\beta_1^* \circ R^m\beta_{2*}(U_2)\right]\right) \neq \\ &\neq R^m\beta_{2*}\left(\left[\beta_2^* \circ R^n\beta_{1*}(U_1)\right] \otimes U_2\right) \end{aligned} \quad (\text{B.5})$$

in general. The solution to this problem is well-known, one has to work in the derived category. That is, the tensor product has to be replaced by the derived tensor product, and we have to take the hypercohomology of the resulting complexes. Fortunately, this is relatively easy for skyscraper sheaves on \mathbb{P}^1 . Their derived tensor product is simply

$$\mathcal{O}_p \otimes_L \mathcal{O}_q = \begin{cases} 0, & p \neq q, \\ \mathcal{O}_p \oplus \mathcal{O}_p[-1], & p = q. \end{cases} \quad (\text{B.6})$$

Therefore, if $p \neq q$ then

$$H^i(X, U) = 0, \quad (\text{B.7})$$

whereas if $p = q$ then

$$\begin{aligned} H^i(X, U) &= H^i\left(R^*\beta_{1*}(U_1) \otimes_L R^*\beta_{2*}(U_2)\right) = \\ &= H^i\left(\mathcal{O}_p[1] \otimes_L \mathcal{O}_q[1]\right) = H^i\left(\mathcal{O}_p[2] \oplus \mathcal{O}_p[1]\right) \simeq \begin{cases} 0 & i = 3, \\ \mathbb{C} & i = 2, \\ \mathbb{C} & i = 1, \\ 0 & i = 0. \end{cases} \end{aligned} \quad (\text{B.8})$$

Notice that we could have used the ordinary tensor product and cohomology as long as we are only computing $H^2(X, U)$. This is precisely what we did in Section 5, and it is justified through the above computation.

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